CYCLICALLY SYMMETRIC PROBLEMS OF HEAT CONDUCTION IN PERFORATED PLATES AND SHELLS WITH HEAT TRANSFER

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The problem of determining the steady-state temperature field is analyzed for the case of plates and hollow shells with a circle of holes and boundary conditions of the third kind.

1. In analyzing the state of stress in thin-walled structural elements under conditions of nonuniform heating, there arises the necessity of determining the temperature field [1, 2]. In this connection, we will consider here a method of analytically determining the temperature field of shells with circular inhomogeneities.

The equation of heat conduction for thin shells, with the median surface referred to curvatures α and β, can be written as [2, 3]:

$$\Delta t+\frac{\partial^2 t}{\partial\gamma^2}+2K\frac{\partial t}{\partial\gamma}=\frac{1}{a^2}\frac{\partial t}{\partial\tau},$$
$$\Delta=\frac{1}{AB}\left[\frac{\partial}{\partial\alpha}\left(\frac{B}{A}\frac{\partial}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(\frac{A}{B}\frac{\partial}{\partial\beta}\right)\right],\quad K=\frac{1}{2}(k_1+k_2). \tag{1}$$

According to [4, 5] and with $K\gamma$ considered negligibly smaller than unity, as in the derivation of Eq. (1), in order to find the mean-over-the-thickness shell temperatures

$$T_1=\frac{1}{2h}\int_{-h}^{h}t\,d\gamma,\quad T_2=\frac{3}{2h^2}\int_{-h}^{h}t\gamma\,d\gamma, \tag{2}$$

needed for determining the temperature moments and the thermal forces, we have the following system of approximate equations

$$h^2\Delta T_1-(\varepsilon_1+K^2h^2)T_1-(\varepsilon_2-Kh)T_2-\frac{h^2}{a^2}\frac{\partial T_1}{\partial\tau}=-(\varepsilon_1t_1+\varepsilon_2t_2),$$
$$h^2\Delta T_2-3\left(1+\varepsilon_1\right)T_2-3(\varepsilon_2-Kh)T_1-\frac{h^2}{a^2}\frac{\partial T_2}{\partial\tau}=-3(\varepsilon_1t_2+\varepsilon_2t_1), \tag{3}$$

where

$$\varepsilon_1=\frac{h}{2}(h_t^++h_t^-),\quad \varepsilon_2=\frac{h}{2}(h_t^+-h_t^-),$$
$$t_1=\frac{1}{2}(t_c^++t_c^-),\quad t_2=\frac{1}{2}(t_c^+-t_c^-). \tag{4}$$

Equations (3), which take into account the heat transfer with the ambient medium according to Newton's law [1], are also based on the assumption that the temperature is linearly distributed over the shell thickness, namely

$$t=T_1+\frac{\gamma}{h}T_2. \tag{5}$$

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Letting K = 0 in (3), we have now the well known equations of heat conduction for thin plates and these equations are also used for approximately determining the temperature field of thin shells, where the effect of the curvature on the temperature distribution is negligible.

If the coupled system (3) has a general solution, then the temperature characteristics T_1 and T_2 can be sought in the form:

$$T_1 = \frac{\lambda_2(\psi_1 - \lambda_1\psi_2)}{\lambda_2 - \lambda_1}, \quad T_2 = \frac{\lambda_2\psi_2 - \psi_1}{\lambda_2 - \lambda_1}. \tag{6}$$

Inserting (6) into (3), we obtain two independent second-order differential equations for the unknown functions ψ_i:

$$\begin{aligned} &\Delta\psi_1 - \mu_1^2\psi_1 - \frac{1}{a^2}\cdot\frac{\partial\psi_1}{\partial\tau} = -f_1, \\ &\Delta\psi_2 - \mu_2^2\psi_2 - \frac{1}{a^2}\cdot\frac{\partial\psi_2}{\partial\tau} = -\frac{1}{\lambda_2} f_2, \end{aligned} \tag{7}$$

where

$$\lambda_{1,2} = 6^{-1}(\varepsilon_2 - Kh)^{-1}\left[(2\varepsilon_1 + 3) \mp \sqrt{(2\varepsilon_1+3)^2 + 12(\varepsilon_2 - Kh)^2}\right],$$

$$\mu_i^2 = h^{-2}\left[\varepsilon_1 + K^2h^2 + 3\lambda_i(\varepsilon_2 - Kh)\right],$$

$$f_i = h^{-2}\left[\varepsilon_1 t_1 + \varepsilon_2 t_2 + 3\lambda_i(\varepsilon_1 t_2 + \varepsilon_2 t_1)\right] \quad (i = 1,\ 2).$$

The boundary conditions and the initial condition for functions T_1 and T_2 are obtained by averaging, in accordance with (2), the respective constraint values of the three-dimensional problem. In this way, the given constraints on the temperature here will be satisfied in the integral sense.

2. Let a thin plate or a hollow shell with a variable temperature and a variable heat transfer be perforated by m identical circular holes whose centers lie on a circle of radius b. We will assume that the centers of these holes are spaced equidistantly and that the boundary conditions are cyclically symmetric.

The steady-state temperature field of a given shell occupying an m-ply connected region (Fig. 1) bounded by the contour $L = L_0 + L_1 + \ldots + L_{m-1}$ will be determined from Eq. (3) with the following boundary conditions on contours L_k ($k = 0, 1, 2, \ldots, m-1$):

$$\begin{aligned} \frac{\partial T_1}{\partial r_k} &= k_t\left[T_1 - T_c^{(1)}\left(\theta_k - 2\pi\frac{k}{m}\right)\right], \\ \frac{\partial T_2}{\partial r_k} &= k_t\left[T_2 - T_c^{(2)}\left(\theta_k - 2\pi\frac{k}{m}\right)\right]. \end{aligned} \tag{8}$$

According to the conditions "at infinity" [7], functions T_1 and T_2 tend toward the given values $T_1^{(\infty)}$ and $T_2^{(\infty)}$ respectively as the distance from a hole approaches infinity ($r \to \infty$).

The general solutions to the homogeneous equations (7) for the m-ply connected region S have the property analogous to that of holomorphic functions [8, 9] and in polar coordinates (r_k, θ_k) they can be written as

$$\psi_i^* = \sum_{p=0}^{\infty} a_p^{(i)} \sum_{k=0}^{m-1} K_p(r_k\mu_i)\cos p\left(\theta_k - \frac{2\pi k}{m}\right), \tag{9}$$

where $K_p(r)$ is the p-th order MacDonald function, $a_p^{(i)}$ are unknown constant coefficients to be determined from the boundary conditions. The particular solutions to these equations will be expressed in the form:

$$\psi_i^0 = \frac{1}{2\pi}\int_s f_i(x_0, y_0) K_0\left[\mu_i\sqrt{(x-x_0)^2 + (y-y_0)^2}\right] dx_0 dy_0, \tag{10}$$

where $K_0(\mu_i r)$ are the fundamental solutions to Eqs. (7) [10].

Inserting the complete solutions

$$\psi_i = \psi_i^* + \psi_i^0 \quad (i = 1,\ 2) \tag{11}$$

into relations (6), we find the integral temperature characteristics T_1 and T_2.

Without detracting from the generality of the solution, we will consider concentrated heating of a shell by the ambient medium:

$$t_c^+ = \delta(x)\,\delta(y), \quad t_c^- = 0,$$

where $\delta(x)$ is the Dirac function.

In this case, by virtue of expression (10), we have

$$\psi_i^0 = q_i K_0(\mu_i r), \quad r = \sqrt{x^2 + y^2}, \tag{12}$$

where

$$q_1 = \frac{\mu(1 + 3\lambda_1)}{4\pi h^2}, \quad q_2 = \frac{\mu(1 + 3\lambda_2)}{4\pi h^2 \lambda_2}, \quad \mu = hh_i^+$$

are Biot numbers and, with (11) and (6), we find

$$T_1 = \frac{\lambda_2}{\lambda_2 - \lambda_1} \sum_{p=0}^{\infty} \left\{ \left[a_p^{(1)} \sum_{k=0}^{m-1} K_p(\mu_1 r_k) - \lambda_1 a_p^{(2)} \sum_{k=0}^{m-1} K_p(\mu_2 r_k) \right] \cos p\left(\theta_k - \frac{2\pi k}{m}\right) + q_1 K_0(\mu_1 r) - \lambda_1 q_2 K_0(\mu_2 r) \right\},$$

$$T_2 = \frac{1}{\lambda_2 - \lambda_1} \sum_{p=0}^{\infty} \left\{ \left[\lambda_2 a_p^{(2)} \sum_{k=0}^{m-1} K_p(\mu_2 r_k) - a_p^{(1)} \sum_{k=0}^{m-1} K_p(\mu_1 r_k) \right] \cos p\left(\theta_k - \frac{2\pi k}{m}\right) + \lambda_2 q_2 K_0(\mu_2 r) - q_1 K_0(\mu_1 r) \right\}. \tag{13}$$

It is not difficult to ascertain that the solutions in form (13) satisfy the conditions of periodicity and "at infinity" [7], so that the boundary conditions need be satisfied on contour L_0 of the first hole only. The boundary conditions on the contours of all other holes are then satisfied automatically.

For convenience, we will write solutions (13) at $r_0 < 2b \sin \pi/m$ in r_0, θ_0 coordinates referred to the first hole. The theorem for combining cylindrical functions [11] and the identities

$$\sum_{k=1}^{m-1} K_{p\pm n}\left(2b\mu_i \sqrt{-i} \sin \frac{\pi k}{m}\right) \sin \pi \left[(n \pm p)\left(\frac{k}{m} + \frac{1}{2}\right) \mp \frac{2kp}{m}\right] \equiv 0,$$

(where p, n, and m are integers) yield

$$T_1 = \frac{\lambda_2}{\lambda_2 - \lambda_1} \sum_{n=0}^{\infty} \left\{ a_n^{(1)} K_n(r_0\mu_1) - \lambda_1 a_n^{(2)} K_n(r_0\mu_2) + \frac{(-1)^n \varepsilon_n}{\pi} \left[q_1 K_n(b\mu_1) I_n(r_0\mu_1) - \lambda_1 q_2 K_n(b\mu_2) I_n(r_0\mu_2) \right] \right.$$
$$\left. + \sum_{p=0}^{\infty} (-1)^p \left[a_p^{(1)} I_n(r_0\mu_1) S^{(1)} - a_p^{(2)} \lambda_1 I_n(r_0\mu_2) S^{(2)} \right] \right\} \cos n\theta_0,$$

$$T_2 = \frac{1}{\lambda_2 - \lambda_1} \sum_{n=0}^{\infty} \left\{ \lambda_2 a_n^{(2)} K_n(r_0\mu_2) - a_n^{(1)} K_n(r_0\mu_1) + \frac{(-1)^n \varepsilon_n}{\pi} \left[q_2 \lambda_2 K_n(b\mu_2) I_n(r_0\mu_2) - q_1 K_n(b\mu_1) I_n(r_0\mu_1) \right] \right.$$
$$\left. + \sum_{p=0}^{\infty} (-1)^p \left[a_p^{(2)} \lambda_2 I_n(r_0\mu_2) S^{(2)} - a_p^{(1)} I_n(r_0\mu_1) S^{(1)} \right] \right\} \cos n\theta_0. \tag{14}$$

Here $I_n(r)$ is a modified Bessel function,

$$S^{(i)} = \sum_{k=1}^{m-1} \left[\cos \pi N K_{p+n}(c_k\mu_i) + \cos \pi M K_{p-n}(c_k\mu_i) \right],$$

$$N = (n + p)\left(\frac{k}{m} + \frac{1}{2}\right) - \frac{2kp}{m}, \quad M = (n - p)\left(\frac{k}{m} + \frac{1}{2}\right) + \frac{2kp}{m},$$

$$c_k = 2b \sin \frac{\pi k}{m}, \quad \varepsilon_n = \begin{cases} \frac{1}{2}, & n = 0, \\ 1, & n \geqslant 1. \end{cases}$$

Inserting the solutions (14) into the boundary conditions (8) for the contour of the first hole, and considering that certain functions can be represented in terms of Fourier series

$$T_c^{(i)} = \sum_{n=0}^{\infty} T_n^{(i)} \cos n\theta_0, \tag{15}$$

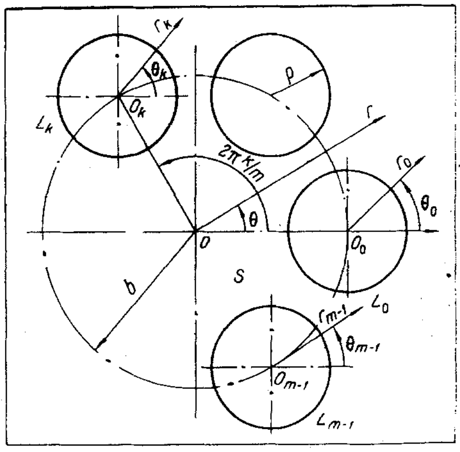

Fig. 1

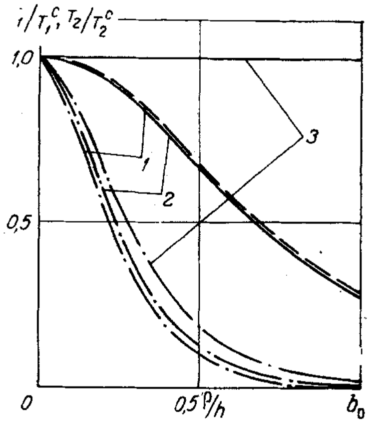

Fig. 2

Fig. 1. Geometry of a perforated shell occupying an m-ply connected region.

Fig. 2. Effect of the distance between holes on the magnitudes of T_1 and T_2 at $\varepsilon_2 = 0$. Solid line and dashed line represent T_1/T_1^c, dashed-dotted lines represent T_2/T_2^c: $\varepsilon_1 = 0.5$ and Kh = 0 (1), $\varepsilon_1 = 0.5$ and Kh = 0.025 (2), $\varepsilon_1 = K = 0$ (3).

we obtain an infinite system of algebraic equations for the constant coefficients $a_n^{(i)}$ and $a_p^{(i)}$:

$$\begin{gathered}
a_n^{(1)}\alpha_n^{(1)} - a_n^{(2)}\lambda_1\alpha_n^{(2)} + \sum_{p=0}^{\infty}(-1)^p\left[a_p^{(1)}\beta_n^{(1)}S^{(1)} - a_p^{(2)}\lambda_1 S^{(2)}\beta_n^{(2)}\right] \\
= \frac{(-1)^n\varepsilon_n}{\pi}\left[\beta_n^{(1)}q_1K_n(b\mu_1) - \lambda_1\beta_n^{(2)}q_2K_n(b\mu_2)\right] + 2k_t\frac{\lambda_2-\lambda_1}{\lambda_2}T_n^{(1)}, \\
-a_n^{(1)}\alpha_n^{(1)} + \lambda_2 a_n^{(2)}\alpha_n^{(2)} + \sum_{p=0}^{\infty}(-1)^p\left[\lambda_2 a_p^{(2)}S^{(1)}\beta_n^{(1)} - a_p^{(1)}S^{(2)}\beta_n^{(2)}\right] \\
= \frac{(-1)^n\varepsilon_n}{\pi}\left[\lambda_2 q_1\beta_n^{(2)}K_n(b\mu_2) - q_1\beta_n^{(1)}K_n(b\mu_1)\right] + 2k_t(\lambda_2-\lambda_1)T_n^{(2)},
\end{gathered} \tag{16}$$

where

$$\alpha_n^{(i)} = \mu_i\left[K_{n-1}(\rho\mu_i) + \frac{2k_t}{\mu_i}K_n(\rho\mu_i) + K_{n+1}(\rho\mu_i)\right],$$

$$\beta_n^{(i)} = \mu_i\left[I_{n-1}(\rho\mu_i) - \frac{2k_t}{\mu_i}I_n(\rho\mu_i) + I_{n+1}(\rho\mu_i)\right].$$

This system of Eqs. (16) can be solved by the reduction method [12]. We note that the right-hand side of Eqs. (16) depends on the particular solutions (10) and on the boundary conditions on the hole contours.

If K = 0 and the coefficients of heat transfer at the shell surfaces are all equal ($\varepsilon_1 \neq 0$, $\varepsilon_2 = 0$), then system (3) breaks up into two independent equations from which each T_i can be determined. The latter can also be determined as the limiting case, however, from relations (6)-(16) with K = 0 and $\varepsilon_2 = 0$. Then $\lambda_1 = 0$, $\lambda_2 = \infty$, and the temperature characteristics

$$T_i = \psi_i.$$

At the same time,

$$q_1 = \frac{\mu}{4\pi h^2}, \quad q_2 = \frac{3\mu}{4\pi h^2}, \quad \mu_1 = \varepsilon_1 h^{-2}, \quad \mu_2 = 3(1+\varepsilon_1)h^{-2}.$$

In this way, we obtain from (14)

$$T_i = \sum_{n=0}^{\infty}\left[a_n^{(i)}K_n(r_0\mu_i) + \sum_{p=0}^{\infty}(-1)^p a_p^{(i)}I_n(r_0\mu_i)S^{(i)} + \frac{(-1)^n\varepsilon_n}{\pi}q_iK_n(b\mu_i)I_n(r_0\mu_i)\right]\cos n\theta_0 \quad (i=1,\ 2), \tag{17}$$

where the unknown coefficients $a_n^{(i)}$ and $a_p^{(i)}$ are determined from the following infinite system of algebraic equations:

$$a_n^{(i)} \alpha_n^{(i)} + \sum_{p=0}^{\infty} (-1)^p a_p^{(i)} \beta_n^{(i)} S^{(i)} = \frac{(-1)^n \varepsilon_n}{\pi} q_i \beta_n^{(i)} K_n (b\mu_i) + 2k_t T_n^{(i)}. \quad (18)$$

We note that, when the solutions to Eqs. (3) with K = 0 are represented according to [13] then the limiting case for $\varepsilon_2 \to 0$ does not follow from the general solution.

3. As an example, we will consider a hollow spherical shell with two identical circular holes whose radius is ρ and on whose contours the conditions

$$T_i = T_i^c = \text{const} \quad (19)$$

are satisfied. The ambient temperature is assumed equal to zero ($t_c^+ = t_c^- = 0$). In this case m = 2, k = 1, $c_k = 2b$, N = M = n, and, with the hole on the right side taken as the first one, relations (9) yield

$$\psi_i = \sum_{n=0}^{\infty} \left\{ a_n^{(i)} K_n (r_0\mu_i) + \sum_{p=0}^{\infty} (-1)^p a_p^{(i)} I_n (r_0\mu_i) S_{np}^{(i)} \right\} \cos n\theta_0. \quad (20)$$

The unknown constant coefficients $a_n^{(i)}$ and $a_p^{(i)}$ are determined from the equations

$$\begin{gathered} a_0^{(i)} K_0 (\rho\mu_i) + \sum_{p=0}^{\infty} (-1)^p a_p^{(i)} S_{0p}^{(i)} I_0 (\rho\mu_i) = F_i^c, \\ a_n^{(i)} K_n (\rho\mu_i) + \sum_{p=0}^{\infty} (-1)^p a_p^{(i)} S_{np}^{(i)} I_n (\rho\mu_i) = 0 \quad (n = 1, 2, \ldots, \infty), \end{gathered} \quad (21)$$

where

$$S_{np}^{(i)} = (-1)^n \left[K_{p+n} (2b\mu_i) + K_{p-n} (2b\mu_i) \right],$$

$$F_1^c = T_1^c + \lambda_1 T_2^c, \quad F_2^c = T_2^c + \frac{1}{\lambda_2} T_1^c.$$

The number of equations in solving system (21) was maintained finite. For a shell with the parameters $\rho = 5h$, $b = 1.25\rho$, and $\varepsilon_1 = 0$, the difference between the values of T_2 calculated at point r = 0, $\theta = 0$ (Fig. 1) with five and with ten equations of system (21) respectively does not exceed 1.5%.

In Fig. 2 is shown the effect which the distance between two holes $b_0 = 2(b-\rho)/h$ has on the magnitude of the dimensionless temperature characteristics T_1/T_1^c and T_2/T_2^c at point r = 0, $\theta = 0$.

The heat transfer from the lateral surfaces and the distance between holes have an appreciable effect on the temperature distribution in a shell. In the case of thin hollow shells, according to the diagram, the curvature has a negligible effect on the temperature field.

NOTATION

r, θ	are the polar coordinates;
r_k, θ_k	are the local polar coordinates;
2h	is the shell thickness;
k_1, k_2	are the principal curvatures of the medium shell surface;
A, B	are the coefficients in the first quadratic representation of the median shell surface;
a^2	is the thermal diffusivity;
h_t^+, h_t^-	are the relative coefficients of heat transfer at the shell surface;
t_c^+, t_c^-	are the ambient temperatures at the shell surface;
$\gamma = \pm h$;	
k_t	is the relative coefficient of heat transfer at a hole contour;
$T_c^{(i)}$	is the integral temperature of the ambient medium;
ρ	is the radius of hole in the shell;
b	is the radius of the holes circle.

LITERATURE CITED

1. A. V. Lykov, Theory of Heat Conduction [in Russian], Izd. Vysshaya Shkola, Moscow (1967).

2. Ya. S. Pidstrigach and S. Ya. Yarema, Thermal Stresses in Shells [in Ukrainian], Izd. Akad. Nauk UkrSSR, Kiev (1961).
3. V. I. Danilovskaya, Izv. Akad. Nauk SSSR, OTN, No. 9 (1957).
4. Ya. S. Pidstrigach, Dokl. Akad. Nauk UkrSSR, No. 5 (1958).
5. Ya. S. Podstrigach, in: Thermal Stresses in Structural Elements [in Russian], Izd. Naukova Dumka, Kiev (1965), Ed. 5.
6. V. V. Bolotin, Prikl. Matem. i Mekhan., 24, No. 2 (1960).
7. R. N. Shvets and V. D. Pavlenko, Fiz.-Khim. Mekhan. Mater., No. 3 (1970).
8. R. C. I. Howland, "Potential functions with periodicity in one coordinate," Proc. Cambridge Phil. Soc., No. 30 (1934).
9. A. N. Guz', Prikl. Mekhan., 4, No. 1 (1968).
10. I. M. Gel'fand and G. E. Shilov, Generalized Functions and Operations on Them [in Russian], Fizmatgiz, Moscow (1959).
11. J. N. Watson, Theory of Bessel Functions [Russian translation], Izd. Inostr. Lit., Moscow (1949), Vol. 1.
12. L. V. Kantorovich and V. I. Krylov, Approximate Methods of Advanced Analysis [in Russian], Fizmatgiz (1962).
13. V. M. Gambara, Abstract Cand. Dissert., L'vov Univ., L'vov (1964).